CARDINAL ARITHMETIC AND REFLECTION THEOREMS FOR THE LINDELÒF DEGREE

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1. CARDINAL FUNCTIONS

Notations:
- CARD: the class of all cardinals
- REG: the class of the regular cardinals
- SING: the class of the singular cardinals

Definition 1.1. [5] Given a space \( X \), \( L(X) \) is the smallest infinite cardinal \( \kappa \) such that every open cover of \( X \) has a subcover of cardinality \( \leq \kappa \).

Definition 1.2. [2] Given a space \( X \), \( ll(X) \) is the smallest infinite cardinal \( \kappa \) such that every open cover of \( X \), linearly ordered by inclusion, has a subcover of cardinality \( \leq \kappa \).

- \( ll(X) \leq L(X) \) for every space \( X \)
- \( X \) is Lindelöf iff \( L(X) = \aleph_0 \)
- \( X \) is linearly Lindelöf iff \( ll(X) = \aleph_0 \)

2. REFLECTION FOR L

Definition 2.1. [6] Let \( \phi \) be a cardinal function, \( \kappa \) be an infinite cardinal, and \( C \) be a class of topological spaces. We say that \( \phi \) reflects \( \kappa \) for the class \( C \), if given a space \( X \) of \( C \) with \( \phi(X) \geq \kappa \), there exists \( Y \subseteq X \) with \( |Y| \leq \kappa \) and \( \phi(Y) \geq \kappa \).

Theorem 2.2. [6]

(1) \( L \) reflects every successor cardinal;
(2) \( L \) reflects every singular strong limit cardinal for the class of Hausdorff spaces;
(3) (GCH + \( \exists \) inaccessible cardinals) \( L \) reflects all cardinals for the class of Hausdorff spaces.

Problem 2.3. Does \( L \) reflect the (strongly or weakly) inaccessible cardinals?

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3. LINDELÖF SPECTRUM

Definition 3.1. [9] Let $X$ be a topological space. For each open cover $C$ of $X$ define

$$m(C) = \min\{|S|: S \text{ is a subcover of } C\}.$$ 

Then define

$$\text{IS}(X) = \{m(C): C \text{ is an open cover of } X\}.$$ 

Note that:
- $X$ is compact iff $\text{IS}(X) \subseteq \omega$
- $X$ is countably compact iff $\aleph_0 \notin \text{IS}(X)$
- $X$ is $[\lambda, \mu]$-compact iff $[\lambda, \mu] \cap \text{IS}(X) = \emptyset$

Proposition 3.2. [9] Given any topological space $X$, we have:

1. if $\lambda \in \text{IS}(X)$ then $\text{cf}(\lambda) \in \text{IS}(X)$;
2. $L(X) = \sup \text{IS}(X) + \aleph_0$;
3. $\text{ll}(X) = \sup(\text{IS}(X) \cap \text{REG}) + \aleph_0$.

Theorem 3.3. [9] Let $\kappa$ be a regular cardinal. Then $L$ reflects $\kappa$ for the class

$$\{X: \text{there is } \xi \in \text{IS}(X) \text{ such that } \xi \geq \kappa\}.$$ 

Reflection of regular cardinals:
- if $L(X) \geq \lambda^+$, then there is $\xi \in \text{IS}(X)$ such that $\xi \geq \lambda^+$
- if $\kappa$ is inaccessible, then there is a space $X$ with $\sup \text{IS}(X) = \kappa$ but $\kappa \notin \text{IS}(X)$

4. COV

Definition 4.1. [13] Definition 5.1 in Chapter II

Given cardinals $\lambda$, $\kappa$, $\theta$ and $\sigma$, $\text{cov}(\lambda, \kappa, \theta, \sigma)$ is the smallest cardinal $\mu$ such that there is a family $\mathcal{P} \subseteq [\lambda]^{<\kappa}$, with $|\mathcal{P}| = \mu$, satisfying:

$$t \in [\lambda]^{<\theta} \Rightarrow \exists \mathcal{Q} \in [\mathcal{P}]^{<\sigma} \left(t \subseteq \bigcup_{A \in \mathcal{Q}} A\right).$$

Note that:
- $\text{cf}(\mu^{<\kappa}, \subseteq) = \text{cov}(\mu, \kappa, \kappa, 2)$
- $\text{cf}(\mu^{\kappa}, \subseteq) = \text{cov}(\mu, \kappa^+, \kappa^+, 2)$

Fact 4.2. [13] Observations 5.2 and 5.3 in Chapter II

1. if $\lambda_1 \leq \lambda_2$, $\kappa_1 \geq \kappa_2$, $\theta_1 \leq \theta_2$, $\sigma_1 \geq \sigma_2$, then
   $$\text{cov}(\lambda_1, \kappa_1, \theta_1, \sigma_1) \leq \text{cov}(\lambda_2, \kappa_2, \theta_2, \sigma_2);$$
2. $\text{cov}(\lambda, \kappa, \theta, \aleph_0) = \text{cov}(\lambda, \kappa, \theta, 2)$ when $\text{cov}(\lambda, \kappa, \theta, \aleph_0) \geq \aleph_0$;
3. if $\lambda \geq \kappa \geq \theta > \sigma = \text{cf}(\sigma)$, then
   $$\text{cov}(\lambda, \kappa, \theta, \sigma) = \sum_{\mu \in [\kappa, \lambda]} \text{cov}(\mu, \mu, \theta, \sigma).$$
5. INACCESSIBLE CARDINALS

**Theorem 5.1.** [9] Let $\kappa$ be an infinite cardinal and $X$ be a topological space. Then $L$ reflects $\kappa$ for $X$, if for every infinite cardinal $\lambda < \kappa$ there is some $\mu \in \mathbb{L}(X)$ such that $\mu > \lambda$ and $\text{cov} (\mu, \mu, \lambda^+, 2) \leq \kappa$.

**Corollary 5.2.** [9] $L$ reflects every strongly inaccessible cardinal.

**Corollary 5.3.**
1. $L$ reflects every successor cardinal and every strongly inaccessible cardinal;
2. $L$ reflects every strong limit cardinal for the class of Hausdorff spaces;
3. (GCH) $L$ reflects all cardinals for the class of Hausdorff spaces.

**Problem 5.4.** [9] Can we show in ZFC that $L$ reflects every weakly inaccessible cardinal?

**Definition 5.5.** [9] $wH$: for every weakly inaccessible cardinal $\kappa$ and every infinite cardinal $\lambda < \kappa$, $|\Theta_{\lambda, \kappa}| < \kappa$, where

$$\Theta_{\lambda, \kappa} = \{ \mu \in \text{CARD} : \lambda < \mu < \kappa, \text{cov} (\mu, \mu, \lambda^+, 2) > \kappa \}.$$

**Theorem 5.6.** [9] (ZFC + $wH$) $L$ reflects every weakly inaccessible cardinal.

**Problem 5.7.** Is $wH$ a theorem in ZFC?

6. CARDINAL ARITHMETIC

Denote by $\text{FIX}$ the class of all fixed points of the $\aleph$-function:

- $\text{FIX} = \{ \lambda \in \text{CARD} : \aleph_\lambda = \lambda \}$
- if $\kappa$ is weakly inaccessible, then $\kappa \in \text{FIX}$ and $|\kappa \cap \text{FIX}| = \kappa$

If $\mu$ is a singular strong limit cardinal, then

$$2^\mu = \text{cov} (\mu, \mu, (\text{cf} (\mu))^+, 2)$$

A short historic revision about exponentiation of singular cardinals:

- (Easton, 1970) exponentiation of regular cardinals
- (Silver, 1974) if $\omega < \text{cf} (\lambda) < \lambda$ and GCH holds below $\lambda$, then $2^\lambda = \lambda^+$
- (Galvin and Hajnal, 1975) if $\lambda = \aleph_\delta$ is singular strong limit, with $\text{cf} (\lambda) > \omega$, then $2^\lambda < \aleph_\delta (2^{\aleph_\delta})^+$
- (Magidor, 1977) A model where GCH holds below $\aleph_\omega$ but $2^{\aleph_\omega} = \aleph_{\omega+2}$
- (Shelah, 1980) Galvin and Hajnal result, for every singular cardinal
- (Shelah, 1994) if $\lambda = \aleph_\delta < \aleph_\lambda$ is singular strong limit, then $2^\lambda < \aleph_{\aleph_\delta}$
(Gitik, 2005) No bound for the first fixed point, see next theorem

**Theorem 6.1.** Suppose that $\kappa$ is a cardinal of cofinality $\omega$ such that for every $\tau < \kappa$ the set \( \{ \alpha < \kappa : o(\alpha) \geq \alpha^{+\tau} \} \) is unbounded in $\kappa$. Then for every $\lambda > \kappa$ there is a forcing extension satisfying the following:

1. $\kappa$ is the first fixed point of the $\aleph$-function,
2. GCH holds below $\kappa$,
3. all the cardinals $\geq \kappa$ are preserved,
4. $2^\kappa \geq \lambda$.

**7. PCF**

**Theorem 7.1.** If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$ then
\[
\text{cf} \left( [\aleph_\delta]^{\mu}, \subseteq \right) < \aleph_{(\delta^{+4})}
\]
for every cardinal $\mu$ with $|\delta| \leq \mu < \aleph_\delta$.

**Corollary 7.2.** If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$ then
\[
\text{cov} \left( \aleph_\delta, \aleph_\delta, \lambda, 2 \right) < \aleph_{(\delta^{+4})}
\]
for every cardinal $\lambda$ with $\lambda < \aleph_\delta$.

**Theorem 7.3.** In the conditions of Definition 5.5, $\Theta_{\lambda, \kappa} \subseteq \text{FIX} \cap \text{SING}$.

**Corollary 7.4.** $L$ reflects every weakly inaccessible cardinal for the class
\[
\{ X : L(X) = \sup (\text{LS}(X) \setminus (\text{FIX} \cap \text{SING})) \}.
\]

**Corollary 7.5.** $L$ reflects every weakly inaccessible cardinal for the class
\[
\{ X : \text{ll}(X) = L(X) \}.
\]

**Problem 7.6.** Is there a consistent example of a space $X$ such that $\text{ll}(X) < \text{cf} (L(X))$?

See also this question on MathOverflow: Linearly Lindelöf spaces with Lindelöf degree of uncountable cofinality

**8. Shelah’s Strong and Weak Hypotheses**

**Definition 8.1.**

1. **SSH**: $\text{pp}(\lambda) = \lambda^+$ for every singular cardinal $\lambda$.
2. **SWH**: For every infinite cardinal $\kappa$,
\[
|\{ \mu \in \text{SING} \cap \kappa : \text{pp}(\mu) \geq \kappa \}| \leq \aleph_0.
\]

SWH is implied by SSH, which is implied by GCH and “$0^\sharp$ does not exist”.

See also List of PCF Hypotheses

When $\aleph_\omega$ is strong limit:
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• (ZFC) $2^{\aleph_\omega} < \aleph_{\omega_4}$
• (ZFC+SWH) $2^{\aleph_\omega} < \aleph_{\omega_4}$
• (ZFC+SSH) $2^{\aleph_\omega} = \aleph_{\omega+1}$

Theorem 8.2. [12, Theorem 6.3(1)] Assuming SSH, for every $\lambda > \kappa$,

$$\text{cov} \left( \lambda, \kappa^+, \kappa^+, 2 \right) = \begin{cases} 
\lambda^+ & \text{if } \text{cf}(\lambda) \leq \kappa \\
\lambda & \text{if } \text{cf}(\lambda) > \kappa.
\end{cases}$$

Theorem 8.3. [9] SSH implies wH.

SSH implies that $\Theta_{\lambda, \kappa} = \emptyset$.

Theorem 6.1 provides a model with $\xi \in \Theta_{\lambda, \kappa}$, where $\kappa$ is a weakly inaccessible cardinal,

$$\xi = \sup \left\{ \aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \ldots \right\}$$

is the first fixed point of the $\aleph$-function, and $\lambda$ is any infinite cardinal with $\lambda < \xi$.

9. SOME NEW HYPOTHESES

Definition 9.1. [9] H1: For every weakly inaccessible cardinal $\kappa$,

$$\left| \left\{ \mu \in \text{SING} \cap \kappa : \text{pp}(\mu) < \text{cov} \left( \mu, \mu, (\text{cf}(\mu))^+, 2 \right) \right\} \right| < \kappa.$$

Definition 9.2. [9] H2: For every weakly inaccessible cardinal $\kappa$,

$$\left| \left\{ \mu \in \text{SING} \cap \kappa : \text{pp}(\mu) \geq \kappa \right\} \right| < \kappa.$$

Some facts about H1:

- $\text{pp}(\mu) \leq \text{cov} \left( \mu, \mu, (\text{cf}(\mu))^+, 2 \right)$ when $\text{cf}(\mu) < \mu$ [10, Proposition 2.7]
- $\text{pp}(\mu) = \text{cov} \left( \mu, \mu, (\text{cf}(\mu))^+, 2 \right)$ when $\text{cf}(\mu) < \mu < \aleph_\mu$ [13, Claim 3.7 of the Chapter IX]
- H1 follows from SSH: under SSH,

$$\text{pp}(\mu) = \text{cov} \left( \mu, \mu, (\text{cf}(\mu))^+, 2 \right) = \mu^+$$

for every singular cardinal $\mu$

Problem 9.3. In ZFC, does the equality

$$\text{pp}(\mu) = \text{cov} \left( \mu, \mu, (\text{cf}(\mu))^+, 2 \right)$$

hold for all singular fixed points of the $\aleph$-function?

See also [13, Problem 14.7(\beta) in “Analytical Guide”] and “cov vs pp” problem

Some facts about H2:

- H2 follows from SWH
- Gitik showed in [4] the consistency of the negation of SWH
Gitik asks (Question 2 in Section 8 of [4]) whether we can have
\[ |\{ \mu \in \text{SING} \cap \kappa : \text{pp}(\mu) > \kappa \} | > \aleph_1 \]
for some cardinal \( \kappa \).

See also Some variants of the Shelah’s Weak Hypothesis.
H1 and H2 may be theorems in ZFC.

10. H1+H2 implies WH

See Possible troubles in Shelah’s book “Cardinal Arithmetic”

Definition 10.1. [9] Given an ordinal \( \alpha \), and any cardinals \( \mu, \eta, \theta \) and \( \sigma \), with \( \eta \geq \theta \), define
\[
s_{\mu,\eta,\theta,\sigma} = \sup \{ \text{cov}(\nu,\nu, (\text{cf}(\nu))^+, \text{cf}(\nu)) : \eta \leq \nu \leq \mu, \sigma \leq \text{cf}(\nu) < \theta \}
\]
and
\[
i_{\mu,\eta,\theta,\sigma}(\alpha) = \begin{cases} 
\mu, & \text{if } \alpha = 0; \\
\sup \{ i_{\mu,\eta,\theta,\sigma} (\beta) : \beta < \alpha \}, & \text{if } \alpha \text{ is a limit ordinal}; \\
i_{\mu,\eta,\theta,\sigma}(\beta) + s_{i_{\mu,\eta,\theta,\sigma}(\beta),\eta,\theta,\sigma}, & \text{if } \alpha = \beta + 1.
\end{cases}
\]

Proposition 10.2. [9] Let \( \mu, \eta, \theta \) and \( \sigma \) be cardinals such that
\[
\mu \geq \eta = \text{cf}(\eta) \geq \theta > \sigma = \text{cf}(\sigma) \geq \aleph_0.
\]
Then,
\[
\text{cov}(\mu,\eta,\theta,\sigma) \leq i_{\mu,\eta,\theta,\sigma}(\omega^2).
\]

Theorem 10.3. [9] H1 + H2 \( \Rightarrow \) WH.

11. Conclusions

If there is a space \( X \) that is a consistent counterexample to the Problem 5.4, then:
- \( L(X) \) is a weakly inaccessible cardinal;
- \( \text{ll}(X) < L(X) = \text{cf}(L(X)) < |X| \);
- there is an infinite cardinal \( \lambda < L(X) \) such that:
  - all elements of \( \text{ls}(X) \setminus \lambda^+ \) are singular fixed points of the \( \aleph \)-function;
  - \( \text{cov}(\mu,\mu,\lambda^+, 2) > L(X) \) for every \( \mu \in \text{ls}(X) \setminus \lambda^+ \);
- the set
  \[
  \{ \mu \in \text{SING} \cap L(X) : \text{cov}(\mu,\mu, (\text{cf}(\mu))^+, 2) \geq L(X) \}
  \]
is unbounded in \( L(X) \).
References


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